# Stochastic integration with respect to the cylindrical Wiener process via regularization

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**Abstract**

Following the ideas of F. Russo and P. Vallois we use the notion of forward integral to introduce a new stochastic integral respect to the cylindrical Winer process. This integral is an extension of the classical integral. As an application, we prove existence of solution of a parabolic stochastic differential partial equation with anticipating stochastic initial date.

**Key words:** Stochastic calculus via regularization, , Russo-Vallois Integrals, cylindrical Wiener process, Stochastic partial differential equation, Parabolic equation.

**MSC2000 subject classification:** 60H05 , 60H15.

## 1 Introduction

Integration via regularization was introduced by F. Russo and P. Vallois ( see [9], [11] and [12]) and it have been studied and developed by many authors.

[1]Research supported by CAPES PNPD N02885/09-3.

They introduced the forward (resp. symmetric) integral $\int_0^T Y d^- X$ (resp. $\int_0^T Y d^\circ X$), the first integral is a generalization of Itô's integral, the second one is an extension of Stratonovich integral, when the integrator $X$ is a semimartingale and the integrand $Y$ is predictable. Moreover, this approach is connected with the Colombeau theory of generalized functions(see [10]). A recent survey on the subject is [14].

In this work we take the approach of regularization to define a new integral respect to the cylindrical Wiener process. This integral is an extension of the classical one. Our method by define the integral via regularization is based in the following representation of the stochastic integral respect to the the cylindrical Winer process

$$\int_0^T g_s dB_s = \sum_{j=1}^{\infty} \int_0^T < g, v_j >_{V_Q} dB(v_j),$$

see sections 3 for details. As an application we study the following SPDE,

$$\begin{cases} u\ dt &= \triangle u\ dt + g(t,x,u)dW_t, \\ u(t,0) &= u(t,1) = 0, \\ u_0 &= f(x,F) \end{cases} \tag{1}$$

where $F$ be a $\mathcal{F}_T$-measure d-dimensional r.v and $W$ is a cylindrical Winer process in $L^2([0,1])$. There is great interest in solving the problem (1) (see for instance [4], [6], [8] , [15] and [16] and references). In this context, and under general assumptions, we show existence for the SPDE (1). The authors do not know any work covering this situation in the literature.

The presentation is organized as follows: in Section 2, we provide some basic concepts on the theory of stochastic integrals with respect to the cylindrical Wiener processes and on stochastic calculus via regularization. In section 3, we introduced a new definition of integral respect to the cylindrical Wiener processes and we show that it is an extension of the integral defined in section

2 . In Section 4, we study the SPDE (1) in the context of this new integration. Moreover, some future works are indicated.

# 2 Preliminaries.

## 2.1 Stochastic integration with respect to the cylindrical Wiener process

Fix a separable Hilbert space $V$ with inner product $< .,. >$. Following [7] and [4], we define the general notion of cylindrical Wiener process in V.

**Definition 2.1** *Let $Q$ be a symmetric (self-adjoint) and non-negative definite bounded linear operator on $V$. A family of random variables $B = \{B_t(h), t \geq 0, h \in V\}$ is a cylindrical Wiener process on $V$ if the following two conditions are fulfilled:*

1. *for any $h \in V$, $\{B_t(h), t \geq 0\}$ defines a Brownian motion with variance $t < Qh, h >$ ,*

2. *for all $s, t \in R+$ and $h, g \in V$*

$$\mathbb{E}(B_s(h)(B_t(g)) = (s \wedge t) < Qh, g >$$

*where $s \wedge t := min(s,t)$. If $Q = Id_V$ is the identity operator in $V$, then $B$ will be called a standard cylindrical Wiener process. We will refer to $Q$ as the covariance of $B$.*

Let $\mathcal{F}_t$ be the $\sigma$-field generated by the random variables $B_s(h), h \in V, 0 \leq\leq t$ and the P-null sets. We define the predictable $\sigma$-field as the $\sigma$-field in $[0,T] \times \Omega$ generated by the sets $(s,t] \times A, A \in \mathcal{F}_s, 0 \leq s < t \leq T$.
We denote by $V_Q$ (the completion of) the Hilbert space V endowed with the inner-product

$$< h,g >_{V_Q} := < Qh, g >, \quad h,g \in V.$$

We can now define the stochastic integral of any predictable square-integrable process with values in $V_Q$ , as follows. Let $\{v_j, j \in \mathbb{N}\}$ be a complete orthonormal basis of the Hilbert space $V_Q$ . For any predictable process $g \in L^2(\Omega \times [0,T], V_Q)$ it turns out that the following series is convergent in $L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the chosen orthonormal system:

$$g \cdot B := \sum_{j=1}^{\infty} \int_0^T < g, v_j >_{V_Q} dB(v_j). \tag{2}$$

We notice that each summand in the above series is a classical Ito integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The stochastic integral $g \cdot B$ is also denoted by $\int_0^T g_s dB_s$ . The independence of the terms in the series (2) leads to the isometry property

$$\mathbb{E}((g \cdot B)^2) = \mathbb{E}((\int_0^T g_s dB_s)^2) = \mathbb{E}(\int_0^T \|g\|_{V_Q}^2 ds).$$

## 2.2 Russo-Vallois Integrals

Let $(X_t)_{t\geq 0}$ be a continuous process and $(Y_t)_{t\geq 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}^+)$, i.e. for any $b > 0$, $\int_o^b |Yt|dt < \infty$ a.s. The generalized stochastic integrals and covariations are defined through a regularization procedure. That is, let $I^-(\epsilon, Y, dX)$ (resp. $I^+(\epsilon, Y, dX), I^0(\epsilon, Y, dX)$ and $C(\epsilon, Y, X)$)) be the $\epsilon-$forward integral (resp. $\epsilon-$backward integral, $\epsilon-$symmetric integral and $\epsilon-$covariation):

$$I^-(\epsilon, Y, dX) = \int_0^t Y_s \frac{(X_{s+\epsilon} - X_s)}{\epsilon} ds \ t \geq 0,$$

$$I^+(\epsilon, Y, dX) = \int_0^t Y_s \frac{(X_s - X_{s-\epsilon})}{\epsilon} ds \ t \geq 0,$$

$$I^0(\epsilon, Y, dX) = \int_0^t Y_s \frac{(X_{s+\epsilon} - X_{s-\epsilon})}{2\epsilon} ds \ t \geq 0,$$

$$C(\epsilon, Y, X) = \int_0^t \frac{(Y_{s+\epsilon} - Y_s)(X_{s+\epsilon} - X_s)}{\epsilon} ds \ t \geq 0,$$

**Definition 2.2** *We set*

$$\textit{Forward integral} \ \int_0^t Y dX^- = \lim_{\epsilon \to 0} I^-(\epsilon, Y, dX)(t),$$

$$\textit{Backward integral} \ \int_0^t Y dX^+ = \lim_{\epsilon \to 0} I^+(\epsilon, Y, dX)(t),$$

$$\textit{Symmetric integral} \ \int_0^t Y dX^0 = \lim_{\epsilon \to 0} I^0(\epsilon, Y, dX)(t),$$

$$\textit{Covariation} \ [X, Y]_t = \lim_{\epsilon \to 0} C(\epsilon, Y, X)(t),$$

*provided the limit exist ucp.*

**Remark 2.1** *a) The links between $[X, Y]$, backward, symmetric and forward integrals are the following*

$$\int_0^t Y dX^\circ = \int_0^t Y dX^- + \frac{1}{2}[X, Y](t).$$

$$[X, Y](t) = \int_0^t Y dX^+ - \int_0^t Y dX^-.$$

*b) If $X$ has locally bounded variation and $Y$ is cad lag then $\int_0^t Y dX^- = \int_0^t Y dX^+$ and it is the pathwise Stieljes integral (see [9]).*

c) *Let $X$ and $Y$ be continuous semimartingales, $H$ be an adapted process. We denote by $\int_0^t HdX$ the Itô integral. Therefore, we have (see [9] )*

$$\int_0^t HdX^- = \int_0^t HdX,$$

*and*

$$[X,Y]_t \text{ is the usual covariation of } X \text{ and } Y.$$

For a recent account in the subject we refer the reader to [14].

# 3 Stochastic integration with respect to the cylindrical Wiener process via regularization

In this section, strongly inspired in the integrals via regularization, we define a new stochastic integral with respect to the cylindrical Wiener process. We will investigate the link between this new integral and the integration defined in section 2.

We shall begin giving the following definition

**Definition 3.1** *Let $g$ be a $V_Q$-valued stochastic process such that the Fourier coefficient $< g, v_j >_{V_Q}$ are finite for all $t$ and $\omega$. Suppose that for all $j \in \mathbb{N}$ there exists*

$$c_j = \int_0^T < g, v_j >_{V_Q} dB_s^-(v_j)$$

*and that $\lim_{m\to\infty} \sum_{j=1}^m c_j$ converges in probability. We define the Russo-Vallois-forward integral $\int_0^T g_s dB_s^-$ by*

$$\int_0^T g_s dB_s^- := \lim_{m\to\infty} \sum_{j=1}^{m} c_j. \tag{3}$$

**Remark 3.1** *1. It is clear of the definition that the Russo-Vallois-forward integral is a linear operator.*

*2. We note that not require any condition of adaptability on the process $g$ through the definition 3.1.*

*3. In the case that $V = \mathbb{R}$ and $Q = Id_{\mathbb{R}}$ the Russo-Vallois-forward integral is equal to forward integral in the sense of the definition 2.2.*

**Proposition 3.1** *For any predictable process $g \in L^2(\Omega \times [0,T], V_Q)$ the integral (2) is equal to the Russo-Vallois-forward integral .*

**Proof:** Since $< g, v_j >_{V_Q}$ is predictable and belongs to $L^2(\Omega \times [0,T])$ we have that

$$c_j = \int_0^T < g, v_j >_{V_Q} dB_s^-(v_j) = \int_0^T < g, v_j >_{V_Q} dB_s(v_j).$$

From the $L^2(\Omega)$-convergence of the series

$$\int_0^T g_s dB_s = \sum_{j=1}^{\infty} \int_0^T < g, v_j >_{V_Q} dB(v_j),$$

it follows that

$$\int_0^T g_s dB_s^- = \int_0^T g_s dB_s.$$

**Remark 3.2** *It is easy to see examples of stochastic process that are Russo-Vallois-forward integral but are not integrable Itô . See remark 14 in [14].*

# 4 Stochastic parabolic equation.

The stochastic partial differential equation type (4) can be analyzed by different approach related with classical deterministic methods, see for example [4], [6], [8], [16] and references. However, in the case that the initial date is anticipating a new solution concept(integral) is need to define. In this direction, we only know the work [15] where the author considerer the finite dimensional initial date.

Let us considerer the following SPDE with homogeneous Dirichlet boundary conditions

$$\begin{cases} u\ dt & = \triangle u\ dt + g(t,x,u)dW_t,\ t \geq 0,\ 0 \leq x \leq 1 \\ u(t,0) & = u(t,1) = 0, \\ u_0 & = f(x,F) \end{cases} \tag{4}$$

under the following hypothesis :

**a**) $W_t$ is a standard cylindrical Wiener process in $L^2([0,1])$ .

**b**) $g$ is continuous function and satisfies

$$|\ g(t,x,y_1) - g(t,x,y_2)| \leq\ C|y_1 - y_2|,$$

**c**) $f(.,z) \in C_0([0,1])$ and satisfies

$$|\ f(x,z_1) - f(x,z_2)| \leq\ C_N|z_1 - z_2|,$$

for $|z_1|, |z_2| \leq N, N > 0$.

**d**) $F$ be a d-dimensional $\mathcal{F}_T$-measure r.v.

**Definition 4.1** *A mild solution of the SPDE (4) is a stochastic process $u$ such that it satisfies*

$$u(t,x) = \int_0^1 G(t-s,x,y)\ f(y,F)\ dy + \int_0^t G(t-s,x,.)\ g(s,.,u)\ dW_s^-.$$

*where $G(t,x,y)$ is the fundamental solution of the heat equation with Dirchlet boundary conditions.*

We recall that $G(t,x,y)$ satisfies

$$\int_0^1 G(t,x,y)^p dy \le C_p\, t^{\frac{1-p}{2}} \textit{ for all } p > 0. \tag{5}$$

**Remark 4.1** *(see [8]) Suppose that* $\{Y_n(z) : z \in \mathbb{R}^d, n \ge 1\}$ *is a sequence of random field such that* $Y_n(z)$ *converge in probability to* $Y(z)$ *as* $n$ *tends to infinity, for each* $z \in \mathbb{R}^d$*. Suppose that*

$$\mathbb{E}|Y(z_1) - Y(z_2)|^p \le C_N |z_1 - z_2|^\alpha.$$

*for* $|z_1|, |z_2| \le N, N > 0$ *and for some constants* $p > 0$ *and* $\alpha > d$*. Then, for any* $d$*-dimensional r.v* $F$ *one has*

$$\lim_{n\to\infty} Y_n(F) = Y(F).$$

*in probability. Moreover, the convergence is* $L^p(\Omega)$ *if* $F$ *is bounded.*

**Theorem 4.1** *Assume that* **a**)*,* **b**)*,* **c**) *and* **d**) *hold . Then there exists a mild solution* $u$ *for the SPDE (4).*

**Proof:** Step 1(Auxiliary problems) We considerer the following family of problems

$$v^z(t,x) = \int_0^1 G(t-s,x,y)\; f(y,z)\; dy + \int_0^t G(t-s,x,.)\; g(s,.,v^z)\; dW_s. \tag{6}$$

We observe that $z \in \mathbb{R}^d$ is a parameter. It is know that there exists an unique solution $v^z(t,x)$ for the problem (6) and it verifies $\sup_{0\le t\le T, 0\le x\le 1} \mathbb{E}[|v^z(t,x)|^2]$ (see for instance p. 142 of [8]).

Step 2 (Estimative of $v^z(t,x)$ ) We claim that $v^z(t,x)$ verifies

$$\mathbb{E}|v^{z_1}(t,x) - v^{z_2}(t,x)|^2 \le C_N |z_1 - z_2|^2. \tag{7}$$

for $|z_1|, |z_2| \leq N, N > 0$. In fact, we have

$$\mathbb{E}|v^{z_1}(t,x) - v^{z_2}(t,x)|^2 \leq C\ (\sup_y \mathbb{E}|f(y,z_1) - f(y,z_2)|)^2$$

$$+ C \int_0^t \int_0^1 G^2(t-s,x,y)\ \mathbb{E}|g(t,y,v^{z_1}) - g(t,y,v^{z_2})|^2\ dyds.$$

By hypothesis **b**), **c**) it follows

$$\mathbb{E}|v^{z_1}(t,x) - v^{z_2}(t,x)|^2 \leq C_N\ |z_1 - z_2|^2$$

$$+ C\ \int_0^t \sup_y \mathbb{E}|v^{z_1}(t,y) - v^{z_2}(t,y)|^2\ \int_0^1 G^2(t-s,x,y)\ dyds.$$

Taking supremo we obtain

$$\sup_x \mathbb{E}|v^{z_1}(t,x) - v^{z_2}(t,x)|^2 \leq C_N\ |z_1 - z_2|^2$$

$$+ C\ \int_0^t \sup_y \mathbb{E}|v^{z_1}(t,y) - v^{z_2}(t,y)|^2\ (t-s)^{-\frac{1}{2}}\ ds.$$

Finally by the Gronwall lemma we get the inequality (7).

Step 3 (Our solution) We shall show that $u(t,x) := v^F(t,x)$ is a mild solution of the problem (4). Let $\{v_j, j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2([0,1])$.
Combining hypothesis **b**), inequality (5) and step 2 we have that

$$\mathbb{E}| < G(t-s,x,.)(g(s,.,v^{z_1}) - g(s,.,v^{z_1})), v_j > |^2 \leq C_N |z_1 - z_2|^2. \qquad (8)$$

and

$$\mathbb{E}| \int_0^t G(t-s,x,.)(g(s,.,v^{z_1}) - g(s,.,v^{z_2}))dW_s|^2 \leq C_N |z_1 - z_2|^2. \qquad (9)$$

for $|z_1|, |z_2| \leq N, N > 0$.
From inequality (8) and by the Russo-Vallois substitution theorems(see for example theorem 1.1 of [9] ) we obtain

$$c_j = \int_0^t < G(t-s,x,.)g(s,.,u), v_j > dB_s^-(v_j) =$$

$$(\int_0^t G(t-s,x,.)g(s,.,v^z), v_j > dB_s(v_j))(F).$$

Moreover, from remark 4.1 and inequality (9) we have $\sum_{j=1}^m c_j$ converges in probability and

$$\lim_{m\to\infty} \sum_{j=1}^m c_j = \int_0^t G(t-s,x,.)\, g(s,.,u)\, dW_s^- = (\int_0^t G(t-s,x,.)\, g(s,.,v^z)\, dW_s)(F).$$

Thus $u(t,x) := v^F(t,x)$ is a mild solution of the SPDE (4).
We end the paper indicating some future works.

**Remark 4.2** *An interesting and potential extension : is the extension for the infinite dimensional stochastic integral in the setup of Da Prato and Zabczyk [5]. This infinite-dimensional stochastic integral can be written as a series of Itô stochastic integrals, see for instance the recent presentation of [4]. We think that can be used a similar scheme to the presented in section 3.*

**Remark 4.3** *It is very well know the relation between the Skorohod-Itô integral and the Wick product (see [10]), this is*

$$\int_0^t X_t dW_s = \int_0^t X_t \diamond W_s ds.$$

*An interesting future work is to study this type relation between integrals via regularization (Forward and symmetric) for the cylindrical Wiener process and distribution products defined via expansion in series(see [1], [2] and [3]). Moreover, we are interested in to study generalized solutions of SPDE drive by a cylindrical Wiener process in a Colombeau type algebra in the new spirit given in [2] and [3].*

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